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SUMMARY

The entire plasma-sheath region is treated by a uniform method for an infinite plane electrode that adjoins a semi-infinite plasma in an electrical field normal to the electrode and a magnetic field parallel to the electrode. The effect of collisions is neglected in calculating the velocity distribution function. Both ions and electrons drift perpendicularly to the electric and magnetic fields, but there is no current to the electrode. A self-consistent potential is calculated, as well as the macroscopic properties of the electron and ion fluids, density, velocity, and pressure tensor components, for a hydrogen plasma in thermal equilibrium. An equilibrium velocity distribution and an isotropic pressure tensor are shown to be established at a distance more than several times the Larmor radius for each species; this minimal distance is reduced for ions in a sheath of large electron surplus. If the plasma density is sufficiently low, the entire transition region is electrically charged, whereas, near an anode in a plasma of higher density, the charge extends only to several times the Debye distance. In the latter case, there is also an exterior region of charge neutrality and potential disturbance that extends to several times the electron Larmor radius for a large anode potential and to several times the ion Larmor radius for other electrode potentials.

INTRODUCTION

When an electrode is placed in a neutral plasma, it tends to disturb the plasma in a way that depends on the electrode potential and the conditions in the plasma. Tonks and Langmuir (ref. 1) showed that the disturbance region may be considered to be divided into a nonneutral region designated as the sheath and a neutral plasma region with a disturbance potential. The purpose of this report is to describe the structure of the steady-state condition of the entire disturbance region adjacent to an infinite plane electrode when a magnetic field exists parallel to the electrode and when the density is sufficiently low that the effect of collisions can be neglected. It is expected that some insight will be provided for the solution of the technically significant case where collisions and time variations occur.

A number of studies have dealt with the electrostatic probe and with the sheath that determines the probe characteristics (see ref. 2 for an extensive bibliography and discussion). The anode probe in a magnetic field is treated by

Bohm, Burhop, and Massey (ref. 3). This treatment is concerned only with the neutral region exterior to the sheath and is appropriate only when the ion Larmor radius is larger than the Debye radius. Bertotti (refs. 4 and 5) discusses the probe in a magnetic field without detailed examination of the sheath other than some general remarks, which are compared herein with the results of the present calculations. The transition region considered herein includes both the region of net electrical charge and the region of disturbance of electrical field by the electrode.

The method used in this study is similar to that of Bernstein and Rabinowitz (ref. 6) in that the entire disturbance region generated by the electrode without a magnetic field is treated as a whole by means of the velocity distribution function and by ignoring the effect of collisions. Hall (ref. 2) points out that this method is suitable only for a probe of limited size, such as the spherical or cylindrical ones considered by Bernstein and Rabinowitz, but is unsuitable for a plane electrode, because a plane electrode that neutralizes impinging particles does not permit the return of attracted particles to the plasma, so that near the electrode the velocity distribution of the attracted species is one sided (no particles proceed away from the electrode). This situation will also persist indefinitely into a collisionless plasma since no mechanism is provided to effect the transition from the one-sided to the full velocity distribution. In the case of the spherical probe, the number of attracted particles that would strike the probe so decreases with distance that a transition to a full distribution is accomplished without collisions.

In case there is a magnetic force field directed parallel to a plane electrode (see fig. 1), a transition to an equilibrium distribution in the plasma is possible without collisions because the motion of the particles normal to the electrode is inhibited. In this situation, however, no electrode current is possible; steady-state currents to the electrode are present only when there are collisions. The effect of a very small collision rate cannot be ignored altogether, because the effect of collisions is cumulative. If the collisions were ignored entirely, the plasma would remain frozen in whatever velocity distribution might be initially imposed. If collisions occur so infrequently that the orbit of the average particle is slowly altered, the motion might be calculated over short periods of time and the slow changes ignored. After a long period of time, the velocity and spatial distributions will have altered substantially from the initial condition to an equilibrium distribution (except for the disturbance created by the electrode).

The assumptions of the problem to be treated are the following:

(1) The plasma and the sheath are at such a low density that collisions have a negligible effect on the velocity distribution at any instant of time. The long-period effect of collisions is accounted for by assuming an equilibrium distribution far from the electrode.

(2) Changes with time are sufficiently slow that at any instant of time there is approximately a steady-state condition.

(3) An infinite plane electrode exists in the z,x plane, and the problem is one dimensional.

(4) The electrode has absorbed or neutralized all particles that may have impinged upon it.

(5) The plasma is neutral far from the electrode.

(6) The plasma as a whole does not drift parallel to the magnetic field.

(7) Thermal and drift velocities are small compared with the speed of light.

DERIVATION OF DISTRIBUTION FUNCTION

Equilibrium Distribution

The electric field $\vec{E}$ is directed normally to the electrode in the plane $y = 0$, so that the electric potential φ is a function of y alone. The magnetic field $\vec{B}$ is taken to be in the direction i_z so that the magnetic potential $\vec{A}$ satisfies

$$\vec{A} = A(y)i_x$$

$$\vec{B} = B(y)i_z$$

$$= -\frac{dA}{dy} i_z$$

(All symbols are defined in appendix A.) Then, in Gaussian units, the Maxwell equations are

$$\frac{d^2\varphi}{dy^2} = 4\pi ec^2(n_e - Z_i n_i) \tag{1}$$

$$\frac{d^2A}{dy^2} = 4\pi e(n_e\overline{v}_{x,e} - Z_i n_i \overline{v}_{x,i}) \tag{2}$$

where

$$\varphi(0) = A(0) = 0$$

$$\lim_{y\to\infty} \frac{A}{y} = -B_0$$

$$\lim_{y\to\infty} \frac{\varphi}{y} = -E_0$$

and e is the proton charge, c is the speed of light, n is the number density, Z is the number of charges per ion, $\overline{v}_x$ is the average velocity in the x-direction, i_x, i_y, and i_z are unit vectors, and the subscripts e and i refer to electrons and ions, respectively. When a particle of mass m moves in a steady electric field, the energy

$$\mathcal{E} \equiv \frac{1}{2} mv^2 + Ze\varphi \tag{3a}$$

is constant. Of the generalized momenta,

$$\left.\begin{aligned} p_x &= mv_x + ZeA \\ p_y &= mv_y \\ p_z &= mv_z \end{aligned}\right\} \tag{3b}$$

p_x and p_z are constants of motion. Then, for a given field, the velocity v_y is obtained from equation (3a) and the integrals of motion:

$$v_y^2 = \frac{2}{m} (\mathcal{E} - Ze\varphi) - \frac{(p_x - ZeA)^2 + p_z^2}{m^2} \tag{4}$$

For the steady, collisionless plasma the velocity distribution function f depends on the variables y, v_x, v_y, and v_z. (See appendix B for conditions under which collisions can be ignored.) When these variables are transformed to the variables $\mathcal{E}$, p_x, v_y, and p_z, the Boltzmann equation may be written

$$\frac{\partial f}{\partial \mathcal{E}} \frac{d\mathcal{E}}{dt} + \frac{\partial f}{\partial p_x} \frac{dp_x}{dt} + \frac{\partial f}{\partial p_z} \frac{dp_z}{dt} + \frac{\partial f}{\partial v_y} \frac{dv_y}{dt} = 0$$

Because $\mathcal{E}$, p_x, and p_z are constants of motion, the solution is

$$f = f(\mathcal{E},p_x,p_z)$$

This solution indicates that the particles will remain permanently in any initial distribution. Actually, the collisions will cause a slow change to an equilibrium distribution; therefore, out of all the possible distributions, it is only this one that is of interest here. For a subsystem of a plasma system, the statistical distribution function is an exponential function of a linear combination of the additive integrals of motion (see ref. 7). For the present case, each particle may be considered to be a subsystem because of the assumed weak interaction between the particles; also, f is the statistical distribution function for a single particle, and the integrals of motion are $\mathcal{E}$, p_x, and p_z. The velocity distribution function for particles in equilibrium and with a negligible collision rate is therefore

$$f = C \exp\left(- \frac{\mathcal{E} - \alpha p_x - \beta p_z}{\theta}\right)$$

where C, α, β, and θ are constants. When the constants are evaluated, θ is found to be kT, where k is the Boltzmann constant and T is the temperature, and α and β are the plasma drift velocities along the x and the z axes, respectively.

Effect of Electrode

The effect of the electrode is assumed to consist in the removal of all particles that would collide with it. Thus, for the sheath,

$$f = CH(p_x, p_z, \mathcal{E}) \exp\left(\frac{\alpha p_x + \beta p_z - \mathcal{E}}{\theta}\right)$$

where $H = 0$ for particles that collide with the electrode and $H = 1$ otherwise.

For any particle with assigned values of $\mathcal{E}$, p_x, and p_z, the value of v_y^2 may be found at any point by equation (4); if $v_y^2 < 0$, the particle cannot reach the point. Admissible particles are limited to those that cannot reach the electrode, that is, those for which

$$0 \gtrless v_y^2(0) = \frac{2}{m}\mathcal{E} - \frac{1}{m^2}\left(p_x^2 + p_z^2\right)$$

The electrode effect is then obtained by identifying $H(U)$ with the Heaviside function:

$$\left.\begin{aligned} H(U) &= 0 \qquad \text{for} \quad U < 0 \\ H(U) &= 1 \qquad \text{for} \quad U \gtrless 0 \end{aligned}\right\} \tag{5}$$

where

$$U \equiv \frac{p_x^2 + p_z^2}{m^2} - 2\frac{\mathcal{E}}{m}$$

The situation is actually somewhat more complex than condition (5) would indicate, and consequently the present formulation has a restricted range of application. The limitation on the class of fields for which the condition $U > 0$ is an appropriate expression of the electrode effect is shown as follows. Equation (4) is first cast into the form

$$v_y^2 = \psi(y, p_x) - U \tag{6}$$

where

$$\psi(y, p_x) \equiv \frac{2Ze}{m}\left(\frac{Ap_x}{m} - \varphi\right) - \frac{Z^2e^2A^2}{m^2}$$

$$\psi(y,v_x) \equiv \frac{2ZeA}{m}\left(v_x - \frac{\varphi}{A} + \frac{ZeA}{2m}\right)$$

Therefore,

$$\lim_{y \to \infty} \psi(y,p_x) = -\infty$$

If the function $\psi(y,p_x)$ has a form as in figure 2(a), the particle will oscillate between the two points A and B as determined by the intersection of $\psi(y,p_x)$ with the level of U. It is clear that a particle with $U = 0$ will reach the electrode, and, if $U < 0$, the particle will be absorbed or neutralized. On the other hand, if the electric and magnetic potentials have appropriate functional dependences on y, there may be such a value of p_x that a negative minimum is attained, such as at one of the points M (fig. 2(b)); then the particle may oscillate between the points C and D with a negative value of U and still not touch the electrode. (Smaller oscillation ranges are possible, of course, with larger values of U.) Such cases are not suitable for use with the condition $U > 0$. When $B \neq 0$ is assumed, the minimum of $\psi(y,p_x)$ is attained for the value of p_x such that

$$p_x - m\frac{d\varphi}{dA} = ZeA$$

The extremum of ψ is then

$$\psi_m = \left(\frac{ZeA}{m}\right)^2 + \frac{2Ze}{m}\left(A\frac{d\varphi}{dA} - \varphi\right)$$

and for $U > 0$ to be suitable, ψ_m must be greater than zero.

The distribution functions are now reformulated as functions of velocity rather than as functions of the integrals of motion. The following dimensionless forms of the variables are used:

$$u_x \equiv \frac{v_x - \alpha}{a}$$

$$u_y \equiv \frac{v_y}{a}$$

$$u_z \equiv \frac{v_z - \beta}{a}$$

$$\nu^2(y,v_x) \equiv (u_x - \omega)\frac{2ZeA}{ma}$$

$$\omega \equiv \frac{\varphi - \alpha A - \dfrac{ZeA^2}{2m}}{Aa}$$

where

$$a^2 \equiv \frac{2\theta}{m}$$

The condition $U > 0$ for admissible particles is also incorporated in f, and in terms of velocities it may be expressed as

$$\frac{2ZeA}{am}(u_x - \omega) - u_y^2 = \nu^2 - u_y^2 > 0 \qquad (7)$$

or, alternatively ($A < 0$, $Z_e = -1$),

$$u_{x,e} > \omega_e - \left(\frac{a_e m_e}{2eA}\right) u_{y,e}^2 > \omega_e$$

$$u_{x,i} < \omega_i + \left(\frac{a_i m_i}{2Z_i eA}\right) u_{y,i}^2 < \omega_i$$

Then

$$f_e = C_e H(u_{x,e} - \omega_e) H\left(\nu_e^2 - u_{y,e}^2\right) \exp\left[\frac{e}{\theta_e}(\varphi - \alpha_e A)\right] \exp\left[-\left(u_x^2 + u_y^2 + u_z^2\right)\right]$$

$$f_i = C_i H(\omega_i - u_{x,i}) H\left(\nu_i^2 - u_{y,i}^2\right) \exp\left[-\frac{Z_i e}{\theta_i}(\varphi - \alpha_i A)\right] \exp\left[-\left(u_x^2 + u_y^2 + u_z^2\right)\right]$$

Appending the conditions on u_x ensures real limits $\pm\nu$ for u_y. In the plasma at $y \to \infty$, $A \to -B_0 y$, $\nu^2(v_{x,y}) \to \infty$, and $H = 1$. Since $u_x^2 + u_y^2 + u_z^2$ is bounded, there is the requirement that $\varphi - \alpha_e A$ and $\varphi - \alpha_i A$ both be bounded in order for f to be nonzero and bounded. From this requirement and from the fact that the electromagnetic field is independent of the species under consideration,

$$\alpha_e = \alpha_i = \alpha$$

and

$$\lim_{y \to \infty} \varphi = \alpha A + \text{const.}$$

With the definitions

$$G \equiv G_w + \frac{e}{\theta_e}(\varphi - \alpha A)$$

$$\tau \equiv \frac{\theta_e}{\theta_i}$$

and G_w defined so that

$$\lim_{y \to \infty} G = 0$$

$$\left.\begin{aligned} f_e &= C_e H(u_{x,e} - \omega_e) H\left(\nu_e^2 - u_{y,e}^2\right) \exp(G) \exp\left[-\left(u_{x,e}^2 + u_{y,e}^2 + u_{z,e}^2\right)\right] \\ f_i &= C_i H(\omega_i - u_{x,i}) H\left(\nu_i^2 - u_{y,i}^2\right) \exp(-Z_i \tau G) \exp\left[-\left(u_{x,i}^2 + u_{y,i}^2 + u_{z,i}^2\right)\right] \end{aligned}\right\} \tag{8}$$

and

$$-\frac{d}{dy}\left(\frac{\theta_e G}{e}\right) = E - \alpha B$$

It will be shown subsequently that α is the plasma drift velocity at $y = \infty$. Therefore, $\theta_e G/e$ is the electric potential relative to the drifting plasma; it is the potential disturbance created by the presence of the electrode and is shown in figure 1 as the difference between the applied and the disturbed potential curves.

Moments of Distribution Function

The constants C, α, and β may be related to plasma properties external to the sheath by evaluation of the moments of the velocity distribution functions. In the limit $y \to \infty$, $H = 1$, $G = 0$, and the plasma values of $n_e \to N$ and $n_i \to N/Z_i$ yield values for C. Also, $\lim_{y \to \infty} \bar{u}_x = 0$ is obtained, from which $\lim_{y \to \infty} \bar{v}_{x,e} = \lim_{y \to \infty} \bar{v}_{x,i} = \alpha$. Then, finally, the constants θ_e and θ_i are related to the temperatures by the Boltzmann constant k by the relation $\theta = kT$.

The particle densities are calculated from zero-order moments of the distribution functions as

$$\left.\begin{aligned} n_e &= \frac{N}{\sqrt{\pi}} e^G \int_{\omega_e}^{\infty} \operatorname{erf} \nu_e e^{-u_x^2} \, du_x \\ n_i &= \frac{N}{Z\sqrt{\pi}} e^{-Z_i \tau G} \int_{-\infty}^{\omega_i} \operatorname{erf} \nu_i e^{-u_x^2} \, du_x \end{aligned}\right\} \tag{9}$$

The other macroscopic averages are obtained from the higher moments. Of these averages, the following simple results are noted:

$$\overline{u}_y = \overline{u}_z = 0$$

$$\overline{u_z^2} = \frac{1}{2}$$

From the moments the macroscopic properties are obtained:

Mass density:

$$\rho_e = m_e n_e; \quad \rho_i = m_i n_i; \quad \rho_m = \rho_e + \rho_i$$

Charge density:

$$\sigma_e = -en_e; \quad \sigma_i = Z_i e n_i; \quad \sigma = \sigma_e + \sigma_i$$

Currents:

$$-en_e\vec{\overline{v}}_e \equiv \vec{j}_e = \vec{J}_e + \alpha\sigma_e i_x; \quad \vec{J}_i + \alpha\sigma_i i_x = Z_i e n_i \vec{\overline{v}}_i = \vec{j}_i$$

where

$$\vec{J}_e = -ea_e n_e u_{x,e} i_x$$

$$\vec{J}_i = Z_i e a_i n_i u_{x,i} i_x$$

$$j \equiv j_e + j_i = J + \alpha\sigma i_x$$

$$J = J_e + J_i$$

Pressure tensors for the ions and electrons are defined as

$$\vec{\vec{p}} = nm\overline{(v - \overline{v})(v - \overline{v})}$$

and yield components

$$p_{xx} = nma^2\left[\overline{u_x^2} - (\overline{u}_x)^2\right] = 2n\theta\left[\overline{u_x^2} - (\overline{u}_x)^2\right]$$

$$p_{yy} = 2n\theta\overline{u_y^2}$$

$$p_{zz} = n\theta$$

$$p_{xy} = p_{yx} = p_{zx} = 0$$

The off-diagonal components are zero because the velocity distribution is an even function of u_y and u_z and the integration is symmetrical. Also, $\overline{u}_y$ and $\overline{u}_z$

are zero for the same reason.

SOLUTION OF EQUATIONS

Change of Variables

Aside from the effect of the electrode in changing the distribution functions f from the Maxwell type, the distribution functions and the densities n_e and n_i depend on the disturbance potential G, which is therefore a more convenient variable for the problem than φ. In addition, the cutoff parameters ω and ν depend on A in the combination given by

$$\zeta \equiv -\frac{eA}{m_e a_e} = -\frac{A}{B_0 L_e} = \int \frac{B}{B_0 L_e}\,dy$$

where L_e $(= a_e m_e/eB_0)$ is the Larmor radius of the electrons. Although ζ is actually a dimensionless magnetic flux, the interpretation of ζ as the approximate distance from the electrode in units of electron Larmor radius will be employed as a more simple and descriptive concept. The system of equations (1) and (2) may also be reduced from fourth to third order by changing the independent variable from y to ζ; the spatial dependence of the solution $y(\zeta)$ may then be found by a subsequent integration. The equations are

$$-\frac{B^2}{B_0^2}\left(\frac{D}{L_e}\right)^2 \frac{d^2G}{d\zeta^2} = \left(1 - \frac{\alpha^2}{c^2} + \frac{a_e\alpha}{2c^2}\frac{dG}{d\zeta}\right)\left(\frac{\sigma}{eN}\right) - \left(\frac{a_e\alpha}{c^2} - \frac{a_e^2}{2c^2}\frac{dG}{d\zeta}\right)\left(\frac{J_x}{eNa_e}\right) \tag{10}$$

$$\frac{d}{d\zeta}\left(\frac{B^2D^2}{B_0^2L_e^2}\right) = \left(\frac{a_e\alpha}{c^2}\right)\left(\frac{\sigma}{eN}\right) + \frac{a_e^2}{c^2}\left(\frac{J_x}{eNa_e}\right) \tag{11}$$

and

$$\frac{d}{d\zeta}\left(\frac{y}{D}\right) = \frac{B_0L_e}{BD}$$

where

$$D = \sqrt{\frac{\theta_e}{4\pi Ne^2c^2}}$$

is the Debye distance, and $L_e^2/D^2 = 8\pi Nm_e c^2/B_0^2$. The method of integration of equation (10) is described in appendix C.

Partial Integration

Equation (11) may be integrated and the system of equations reduced to second order. For this purpose the momentum equation for the mixture of ions and electrons in a steady-state condition is utilized:

$$\nabla \cdot (\rho_m \vec{\bar{v}}\vec{\bar{v}}) = -\nabla \cdot \vec{\bar{p}} + \sigma\vec{E} + \vec{j} \times \vec{B}$$

where

$$\rho_m \vec{\bar{v}} \equiv \rho_e \vec{\bar{v}}_e + \rho_i \vec{\bar{v}}_i$$

For the case at hand,

$$\vec{\bar{v}}_e = i_x \bar{v}_e$$

$$\vec{\bar{v}}_i = i_x \vec{\bar{v}}_i$$

$$\vec{E} = i_y E$$

$$\vec{B} = i_z B$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} = 0$$

The relations

$$\nabla \cdot \vec{E} = 4\pi c^2 \sigma$$

$$\nabla \times \vec{B} = 4\pi \vec{j}$$

inserted into the momentum equation, when integrated, yield

$$p_{yy} + \frac{B^2}{8\pi} - \frac{E^2}{8\pi c^2} = p_0 + \frac{B_0^2}{8\pi}\left(1 - \frac{\alpha^2}{c^2}\right)$$

where

$$p_0 \equiv p_{0,i} + p_{0,e} = N\theta_e\left(1 + \frac{1}{Z\tau}\right)$$

and p_{yy} is calculated from the sum of the second moments of the distribution functions of the ions and the electrons, that is,

$$p_{yy} = 2N\theta_e\left[\left(\frac{n_i}{N\tau}\right)\overline{u^2_{y,i}} + \left(\frac{n_e}{N}\right)\overline{u^2_{y,e}}\right]$$

In order to put this result in terms of the variables being used,

$$E = \left(\alpha - \frac{a_e}{2}\frac{dG}{d\zeta}\right)B$$

$$2N\theta_e = \frac{B_0^2 L_e^2 a_e^2}{8\pi D^2 c^2}$$

are employed, and the result is

$$\frac{B^2}{B_0^2}\left[1 - \left(\frac{\alpha}{c} - \frac{1}{2}\frac{a_e}{c}\frac{dG}{d\zeta}\right)^2\right] = 1 - \frac{\alpha^2}{c^2} + \frac{L_e^2}{D^2}\frac{a_e^2}{c^2}\left[\frac{1}{2}\left(1 + \frac{1}{Z\tau}\right) - \frac{n_i}{N\tau}\overline{u^2_{y,i}} - \frac{n_e}{N}\overline{u^2_{y,e}}\right] \tag{13}$$

which replaces equation (11). Because E/B is approximately equal to the drift velocity $\overline{v}_x$, equation (12) shows that the electric tension $E^2/8\pi c^2$ is always small compared with the magnetic pressure $B^2/8\pi$. The magnetic and kinetic pressures are therefore always in balance, and the maximum change in magnetic pressure is approximately equal to the kinetic pressure of the plasma p_0.

RESULTS

The foregoing equations were integrated with consistent values for density and current; the results of the calculations are given in figure 3 as curves of the potential disturbance function G, the density ratios n_e/N and n_i/N, and the ion and electron pressure ratios p_{xx}/p_0 and p_{yy}/p_0 plotted as functions of ζ ($\approx y/L_e$). The pressure ratio p_{zz}/p_0 is indicated to be equal to n/N. The calculations were made for $G_w = \pm 1$ and ± 5, $D/L_e = 100$, 2, 0.5, and 0, and several values of α/c and a/c. Since the effects of α and a are very small, these values are not shown. In the sequence the curves for $D/L_e = 0$, $G_w = \pm 1$ are omitted because they are very much like the curves for $D/L_e = 0.5$. The method of integration used was unsuitable for the cases $D/L_e = 0.5$, $G_w = \pm 5$, and no results were obtained. Presumably these curves would be similar to the curves for $D/L_e = 0$, $G_w = \pm 5$. Because the coefficient of the highest derivative is small in these cases, standard integration techniques are not applicable. Also omitted is the curve for $G_w = -5$, $D/L_e = 0$. This curve violated condition (7) over a considerable region, so that the effect of the electrode is not accurately stated in condition (5). The same violation occurred for $G_w = -5$, $D/L_e = 0.5$ but extended over a small region ($\zeta < 4$) where the density is low and presumably the effect is not large.

Although α has an insignificant effect on the disturbance potential G, its effect on the actual electric potential φ relative to the electrode can be seen from the relations

$$\frac{e}{\theta_e} \varphi = G - G_W - 2 \frac{\alpha}{a_e} \zeta$$

$$\frac{d}{d\zeta} \left(\frac{e\varphi}{\theta_e}\right) = \frac{dG}{d\zeta} - \frac{2\alpha}{a_e}$$

When α/a_e is sufficiently large, the effect of the externally applied field overshadows that of the disturbance field.

Isotropy and Equilibrium

All the plots show some region in which either the electron or the ion pressure tensor is not isotropic. This deviation from isotropy of the pressure tensor is a measure of the electrode effect, since it is caused by the selective removal of particles with prohibited values of the momenta p_x and p_y normal to the magnetic force lines and by the absence of such selection on the parallel component p_z. This same process causes the deviation of the velocity distribution function from the equilibrium (Maxwell) distribution and reduces the number density below the equilibrium value Ne^G for electrons or $(N/Z)e^{-Z\tau G}$ for ions. Consequently, the nonisotropy of the pressure tensor is considered a measure of deviation from equilibrium. In all cases equilibrium is established at a distance of 4 to 5 Larmor radii appropriate to the species.

Pressure isotropy (equilibrium) of the repelled species is enhanced by a large electrode potential. This phenomenon results from the variation of the electrical field intensity (curvature of the G curve). A uniform field has no effect on ion collision with the anode, because a Lorentz transformation of the electromagnetic field to a coordinate system moving with the local drift velocity removes the electric field, and the ions orbit in circular paths of a size independent of the field strength. If the electric field is nonuniform, it cannot be eliminated by transformation; consequently, the orbital path is modified from the circular shape in such a way that repelled particles require higher energy to reach the electrode than in a uniform field. Thus, when the field strength is highly nonuniform, a very small proportion of the population of the repelled species is removed, and the distribution closely approximates the Maxwell distribution with an isotropic pressure tensor. The trend to equilibrium of electrons is smaller than that of ions because of the reduced Larmor radius and the smaller variation of electric field in the orbit. These characteristics may be observed by comparing figures 3(a) to (d), where the sheath is calculated for various electrode potentials. Figures 3(e) to (h) show a similar change in approach to equilibrium.

Scale of Region of Potential Variation

The extent of the region in which the potential of electrical disturbance varies substantially is shown by Bertotti (ref. 4) to be of the order of the sum of the Debye distance and the Larmor radius of the attracted species, on the basis that the electrode influences the density distribution of the attracted species to distances of the order of the Larmor radius, whereas the repelled species is in equilibrium. The distance scale of Poisson's equation is the Debye distance in that charge disturbances influence the potential as far as the Debye distance; beyond this range shielding by other charges takes place. All the calculations substantiate this conclusion (fig. 3). There is an additional smaller variation of potential well in a region much larger than this one in cases of low-voltage anodes with weak magnetic fields (figs. 3(f) and (k)), where the electrode potential is not sufficiently large to effect an equilibrium distribution of the repelled species (ions); nonisotropy of the ion pressure tensor indicates this situation.

Regions of Neutrality and Charge

All calculations show the attainment of charge neutrality in regions farther from the electrode than several times the Debye length D ($y \gtrsim \approx 5D$); cathode sheaths in a weak magnetic field (figs. 3(g), (h), and (j)) exhibit only small charges even closer to the electrode. Poisson's equation (eq. (10)) is directly related to this situation in that the characteristic length for variation of potential is the Debye length; for $y > 5D$ the derivative is small and a solution of the equation is the plasma condition of neutrality ($n_e = Zn_i$).

The potential variation in the neutral region may be correlated with the Larmor radius and the electrode potential by consideration of equations (9) for density, where K represents the integral factor:

$$\left.\begin{aligned} \frac{n_e}{N} &= K_e e^{G} \\ \frac{Zn_i}{N} &= K_i e^{-Z\tau G} \end{aligned}\right\} \tag{14}$$

The integrals K are less than 1.0 by an amount that represents the fraction of the population that has been removed by the electrode. This interpretation then indicates that $K \rightarrow 1$ at a distance of several times the Larmor radius since only the extremely high-energy particles in a Maxwell distribution could reach the electrode and be removed. This distance will be taken to be $5L$. At smaller distances $K \rightarrow 1$ also for a strongly repelled species. In general, $K_e \gtrless K_i$ because of the difference in Larmor radii, but the case of an anode with a large potential is different in that $K_i = 1$ (except possibly for a very small region near the electrode), and in this case $K_e < K_i = 1$ in the region $y < 5L_e$. The neutrality condition

$$\exp[(1 + \tau Z)G] = \frac{K_i}{K_e}$$

can be used with these interpretations to indicate when $G > 0$ ($K_i > K_e$) or $G < 0$ ($K_i < K_e$).

In weak plasmas the neutral condition is attained only for $y > 5D > 5L_i$. Neutrality thus implies $K_e = K_i = 1$ and, therefore, $G = 0$, which also implies

$n_e = Zn_i = N$; that is, the plasma is undisturbed altogether in a neutral region (figs. 3(a) to (d)).

If the anode potential is large ($G_w \gtrsim +5$), $K_i \approx 1$, and, therefore, when $D < L_e$ (dense plasma), then $K_e < 1$ in the region $5D < y < 5L_e$. The neutrality condition then indicates that $G > 0$ (fig. 3(k)). For the region $5L_e < y$, $K_e = 1$ and $G = 0$ in any neutral region for both medium and dense plasmas (figs. 3(e) and (k)).

For the cases where $G_w = \pm 1$ or -5 the condition $K_i < K_e \lessapprox 1$ and the neutrality condition yield $G < 0$ where $5D < y < 5L_i$. In the range $y > 5L_i$, $K_i = K_e = 1$; here the neutrality condition indicates the absence of any electrode disturbance at all (figs. 3(f) to (j)).

The anode sheaths exhibit stronger charges than the corresponding cathode sheaths (compare fig. 3(b) with (c), (e) with (h), (f) with (g), and (i) with (j)). This effect is a consequence of the large Larmor radius of the ions, which extends the region of ion population depletion by the electrode. In a region of electron surplus near an anode, the ion depletion thus increases the sheath charge, whereas near a cathode the densities of the two species are made more nearly equal. The effect is large enough to effect a condition of approximate neutrality in the inner region ($y < 5D$) of the sheath in weaker magnetic fields (figs. 3(g), (h), and (j)) where the ion Larmor radius is larger than the Debye distance.

SUMMARY OF RESULTS

When collisions are ignored, calculations of the sheath between a plane electrode and a hydrogen plasma in thermal equilibrium in the presence of a magnetic field parallel to the electrode show that

1. The dimensionless disturbance potential relative to the moving plasma, the densities, and the pressure tensor components (ratio to undisturbed-plasma value) are all independent of the plasma temperature and drift velocity; they depend only on dimensionless flux (approximately the ratio of distance to Larmor radius), ratio of Larmor radius to Debye distance, and ratio of electrode potential energy to thermal energy.

2. The electrode effect on velocity distribution and pressure tensor components extends to a distance of five times the Larmor radius, except that the variation of electric field intensity (which is large with a substantially charged sheath) results in a reduced electrode effect on the repelled species; this electrode effect is less pronounced with ions near anodes than with electrons near cathodes.

3. The plasma condition of neutrality is not satisfied in the transition region for rare plasmas ($D > L_i$) nor in the region $y < 5D$ near anodes in medium-density plasmas ($L_e < D < L_i$), where D is the Debye distance, L_e is the electron Larmor radius, L_i is the ion Larmor radius, and y is the distance normal to the electrode. It is satisfied in all cases for $y > 5D$ and approximately so for all locations near cathodes in medium or dense plasmas ($D < L_i$).

4. In a region of neutrality the potential is always negative in a region extending to approximately $5L_i$, except that, if the anode potential is large (five times the thermal energy), the potential is positive in the region $y < 5L_e$.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, July 5, 1963

APPENDIX A

SYMBOLS

A	magnetic potential $(B = -dA/dy;\ A(0) = 0)$
a	$\sqrt{2kT/m} = \sqrt{2\theta/m}$
B	magnetic field intensity
b	constant, approximation for $\sqrt{\lambda}$ (see eq. (C3))
c	speed of light
D	Debye distance; $D = \sqrt{kT_e/4\pi Ne^2c^2}$
E	electric field intensity
$\mathcal{E}$	energy of particle
e	proton charge
f	Boltzmann distribution function
G	dimensionless potential relative to plasma; $G \equiv G_w + \frac{e}{\theta_e}(\varphi - \alpha A)$
g	correction to the approximation $\Gamma \approx G$; $g \equiv G - \Gamma$
H	electrode effect function; $H = 1$ for permissible particles and $H = 0$ for disallowed particles
i_x, i_y, i_z	unit vectors in direction of x, y, or z variation
J	electric current relative to plasma; $J = j - \alpha\sigma\nabla x$
j	electric current relative to electrode
K	electrode effect on density depletion (eq. (14))
k	Boltzmann constant
L	Larmor radius of electrons or ions; $L = a/\omega_c$
m	mass of particle
N	number density of electrons in plasma
n	number density of particles in sheath

p	$(p_{xx} + p_{yy} + p_{zz})/3$
p_x,p_y,p_z	components of generalized momentum vector
p_{xx},p_{yy},p_{zz}	components of pressure tensor
S	right side of eq. (10) with σ/eN omitted (see appendix C)
T	temperature of electrons or ions
t	time
U	$\left[\left(p_x^2 + p_z^2\right)/m^2\right] - 2\mathcal{E}/m$
u_x,u_y,u_z	nondimensional velocity components relative to plasma; $u_x = (v_x - \alpha)/a$; $u_y = v_y/a$; $u_z = (v_z - \beta)/a$
v	velocity of particle
$\overline{v}_e,\overline{v}_i$	average velocity of electrons or ions
$\overline{v}_m$	velocity of ion-electron mixture; $\overline{v}_m = \frac{\rho_e}{\rho_m}\overline{v}_e + \frac{\rho_i}{\rho_m}\overline{v}_i$
x,y,z	coordinates
Z	number of fundamental charge units per ion ($Z_e = -1$)
α	constant (drift velocity of plasma, E_0/B_0)
β	constant (drift velocity in z-direction, assumed to be zero)
Γ	approximation for G in iterative process for finding G
δ	$-[(\sigma/eN) + S]B_0^2D^2/B^2L_e^2$ (see eq. (C3))
ϵ	error in eq. (10) when G is replaced by Γ (see eq. (C1))
ζ	$-A/B_0L_e = -Ae/m_ea_e$; $\zeta \approx y/L_e$
θ	constant (shown to be kT_e or kT_i)
Λ	collision parameter (appendix B)
λ	$-\left(B_0^2L_e^2/B^2D^2\right)[\partial(\sigma/eN)/\partial G + \partial S/\partial G]$ (eq. (C2))
ν	electrode cutoff for u_y integration; $\nu \equiv \sqrt{\psi/a^2}$
ν_c	collision frequency per particle
ρ_e,ρ_i	mass density of electrons or ions

ρ_m	mass density of electron-ion mixture; $\rho_m = m_e n_e + m_i n_i$
$\sigma, \sigma_e, \sigma_i$	electric charge density of electron-ion mixture or of either component
τ	$T_e/T_i = \theta_e/\theta_i$
φ	electric potential function $[\varphi(0) = \varphi_w = 0;\ E = -d\varphi/dy]$
ψ	$\frac{2Ze}{m}\left(\frac{Ap_x}{m} - \varphi\right) - \frac{Z^2e^2A^2}{m^2}$
ψ_m	stationary minimum value of ψ
ω	electrode cutoff for u_x integration; $\omega = [\varphi - \alpha A - (ZeA^2/2m)]/Aa$
ω_c	cyclotron frequency; $\omega_c = eB_0/m_e$ or ZeB_0/m_i

Subscripts:

e	electrons
i	ions
w	value at electrode $(y = 0)$
x,y,z	component appropriate to x-, y-, or z-axis direction
0	value in plasma $(y = \infty)$

Superscripts:

$(\bar{\ })$	velocity average
$(\vec{\ })$	vector

APPENDIX B

CONDITIONS FOR WHICH COLLISIONS CAN BE IGNORED

A limitation of the theory arises from the assumption that over a short period of time the effect of collisions on the distribution functions is small. Quantitatively this assumption is expressed by $\nu_c/\omega_c \ll 1$, where ν_c is the scattering collision frequency and ω_c is the cyclotron or Larmor frequency. For consideration of order of magnitude, electron shielding alone is assumed, and conditions are calculated in the plasma rather than in the sheath. Because of the large Larmor radius, ion-ion scattering gives the largest value of ν_c/ω_c. Then, from reference 8 $\nu_{c,i}$ is obtained, from which the result for hydrogen is

$$1 \gg \left(\frac{\nu_c}{\omega_c}\right)_{ii} \approx 0.71\left[\left(\frac{2}{3}\right)1836\right]^{1/2}\left(\frac{L_e \log \Lambda}{D\Lambda}\right)$$

where

$$\Lambda \equiv \frac{3(kT)^{3/2}}{e^2c^2(4\pi Ne^2c^2)^{1/2}}$$

The condition for ignoring collisions is

$$\left(\frac{8\pi m_e Nc^2}{B_0^2}\right)^{1/2} = \frac{L_e}{D} \approx 0.04\left(\frac{\Lambda}{\log \Lambda}\right)\left(\frac{\nu_c}{\omega_c}\right)_{ii} \ll 0.04\left(\frac{\Lambda}{\log \Lambda}\right)$$

which is equivalent to a minimum magnetic field intensity. This limitation is very weak for high temperature and low density and very severe for low temperature and high density. At $T = 1000^\circ$ K and $N = 10^{12}$ per cubic centimeter, for example,

$$\frac{L_e}{D} \ll 2$$

APPENDIX C

METHOD OF INTEGRATION

The system of equations (10) and (11) embodied difficulties peculiar to the nonlinearity of the problem, the two-point boundary conditions, and the infinite range of the independent variable. The method used was one of successive approximations with linearized approximations for the corrections. With equation (10) written in shorter notation,

$$-\frac{D^2}{L_e^2}\frac{B^2}{B_0^2}\frac{d^2G}{d\zeta^2} = \frac{\sigma}{eN} + S \equiv \frac{\sigma}{eN} + \left(\frac{a_e}{2c}\frac{dG}{d\zeta} - \frac{\alpha}{c}\right)\left(\frac{a_e}{c}\frac{J_x}{eNa_e} + \frac{\alpha}{c}\frac{\sigma}{eN}\right)$$

a function $\Gamma(\zeta)$ is assumed to be an approximation for $G(\zeta)$, and the dependence of S on $dG/d\zeta$ is neglected because a^2/c^2 and $a\alpha/c^2$ are small. The correction $g \equiv G - \Gamma$ is assumed small, so that variations in G will cause changes in σ and J_x, which can be approximated by

$$\sigma(G,\zeta) = \sigma(\Gamma,\zeta) + g\left(\frac{\partial\sigma}{\partial G}\right)_{G=\Gamma}$$

$$J(G,\zeta) = J(\Gamma,\zeta) + g\left(\frac{\partial J}{\partial G}\right)_{G=\Gamma}$$

If the error in the differential equation is ϵ when Γ is used,

$$\epsilon \equiv -\left(\frac{B_0^2L_e^2}{B^2D^2}\right)\left[\frac{\sigma(\Gamma,\zeta)}{eN} + S(\Gamma,\zeta)\right] - \frac{d^2\Gamma}{d\zeta^2}$$

$$\frac{d^2g}{d\zeta^2} = \lambda g + \epsilon \tag{C1}$$

where

$$\lambda \equiv -\left(\frac{B_0^2L_e^2}{B^2D^2}\right)\left[\frac{\partial}{\partial G}\left(\frac{\sigma}{eN}\right) + \frac{\partial S}{\partial G}\right] \tag{C2}$$

When $\Gamma(0) = G_w$ and $\Gamma(\infty) = 0$,

$$g(0) = g(\infty) = 0$$

Explicit expressions for the derivatives are

$$-\frac{\partial}{\partial G}\left(\frac{\sigma}{eN}\right) = \frac{n_e}{N} + Z\tau\left(\frac{Zn_i}{N}\right) + \frac{1}{\zeta}\left(\frac{n_e}{N}\,\overline{u}_{x,e} + \sqrt{\frac{\tau m_i}{m_e}}\,\frac{n_i Z}{N}\,\overline{u}_{x,i}\right)$$

$$\frac{\partial}{\partial G}\left(\frac{J}{N_e a_e}\right) = -\frac{n_e}{N}\,\overline{u}_{x,e} - Z\sqrt{\frac{\tau m_e}{m_i}}\,\frac{Zn_i}{N}\,\overline{u}_{x,i} + \frac{1}{2\zeta}\left(-\frac{\sigma}{N_e} + \overline{u^2_{x,e}} - \overline{u^2_{x,i}}\right)$$

Because of the complicated form of the coefficient λ, equation (C1) is not amenable to direct solution for a self-consistent electrical field. A method of successive approximations was used, in which for each approximation a constant value of λ was assumed. Then the correction g is obtained from equation (C1) with the following result:

$$G = g + \Gamma = \frac{e^{b\zeta}}{2b}\int_{\infty}^{\zeta}(\delta - b^2\Gamma)e^{-bx}\,dx - \frac{e^{-b\zeta}}{2b}\int_{0}^{\zeta}(\delta - b^2\Gamma)e^{bx}\,dx$$

$$+ \frac{e^{-b\zeta}}{2b}\int_{0}^{\infty}(\delta - b^2\Gamma)e^{-bx}\,dx + G_w e^{-b\zeta} \tag{C3}$$

where

$$b \equiv \sqrt{\lambda}$$

and

$$\delta \equiv \frac{-\left(\frac{\sigma}{eN} + S\right)B_0^2 D^2}{B^2 L_e^2}$$

For $L_e \ll D$ and the initial approximation $\Gamma = G_w e^{-b\zeta}$, convergence was very rapid (three iterations), but convergence was slow for weak fields ($L_e = O(D)$). In these cases convergence was more rapid for large values of ζ; the rate of convergence near the wall was improved by occasional use of values of λ appropriate for smaller ζ, although the usual procedure was to use $\lambda = (1 + Z\tau)L_e/D$, which is appropriate to large ζ. After each iteration for G, the function B^2/B_0^2 was reevaluated from equation (11), although equation (12) might just as well have been used. This method was found to be unsuitable for values of $G_w > 10$ or $G_w < -10$.

In the limiting case in which the plasma density is very large or the magnetic field weak, the ratio D/L_e becomes vanishingly small. Then the differential equations (10) and (11) are simplified by ignoring the second derivative terms. For this system there is the solution $\sigma = 0$ or $n_e = Zn_i$, which is known as the plasma equation. The plasma equation was checked by the solutions for the cases $D/L_e = 1/2$, $G_w = 1, -1$ by means of integration of the differential equation. The results showed a negligibly small net space charge.

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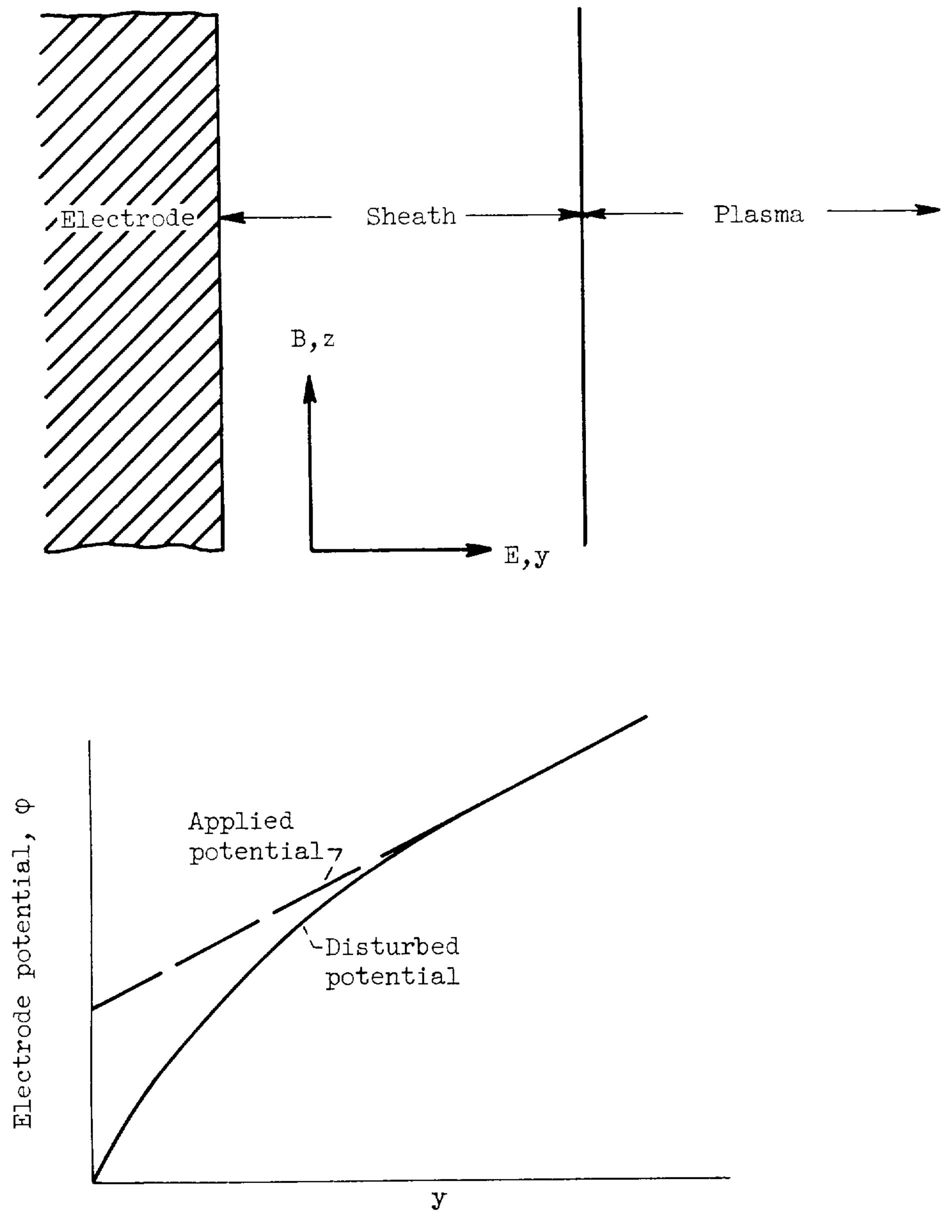
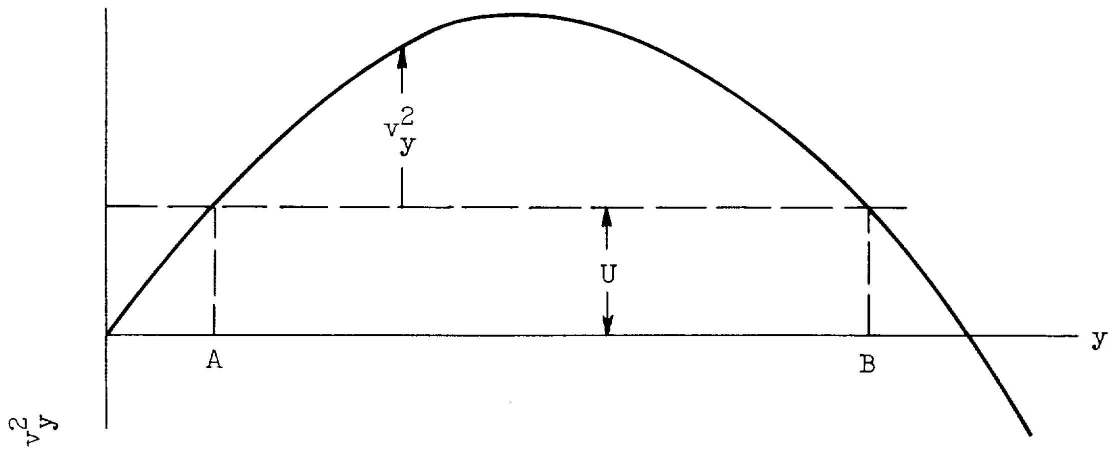
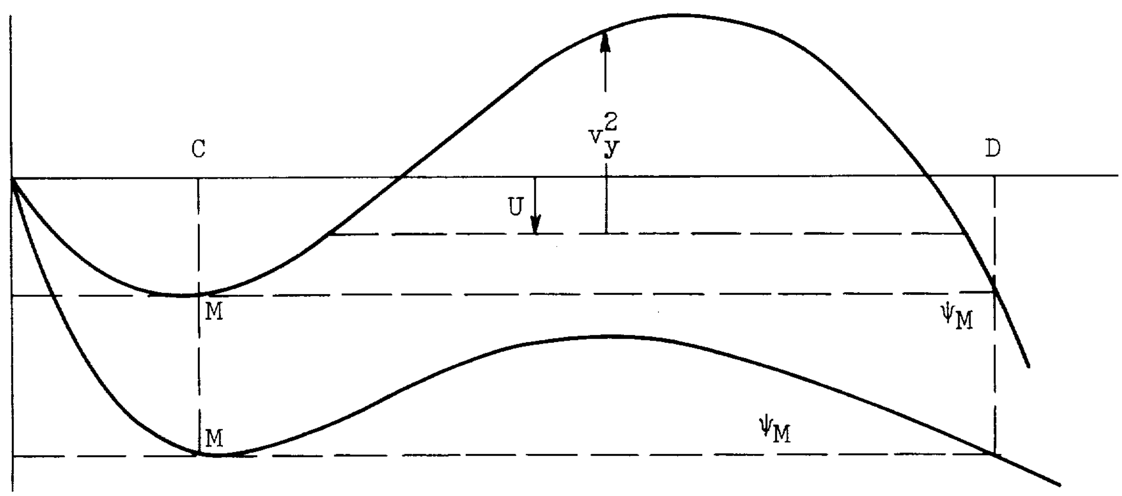


Figure 1. - Configuration and conventions.

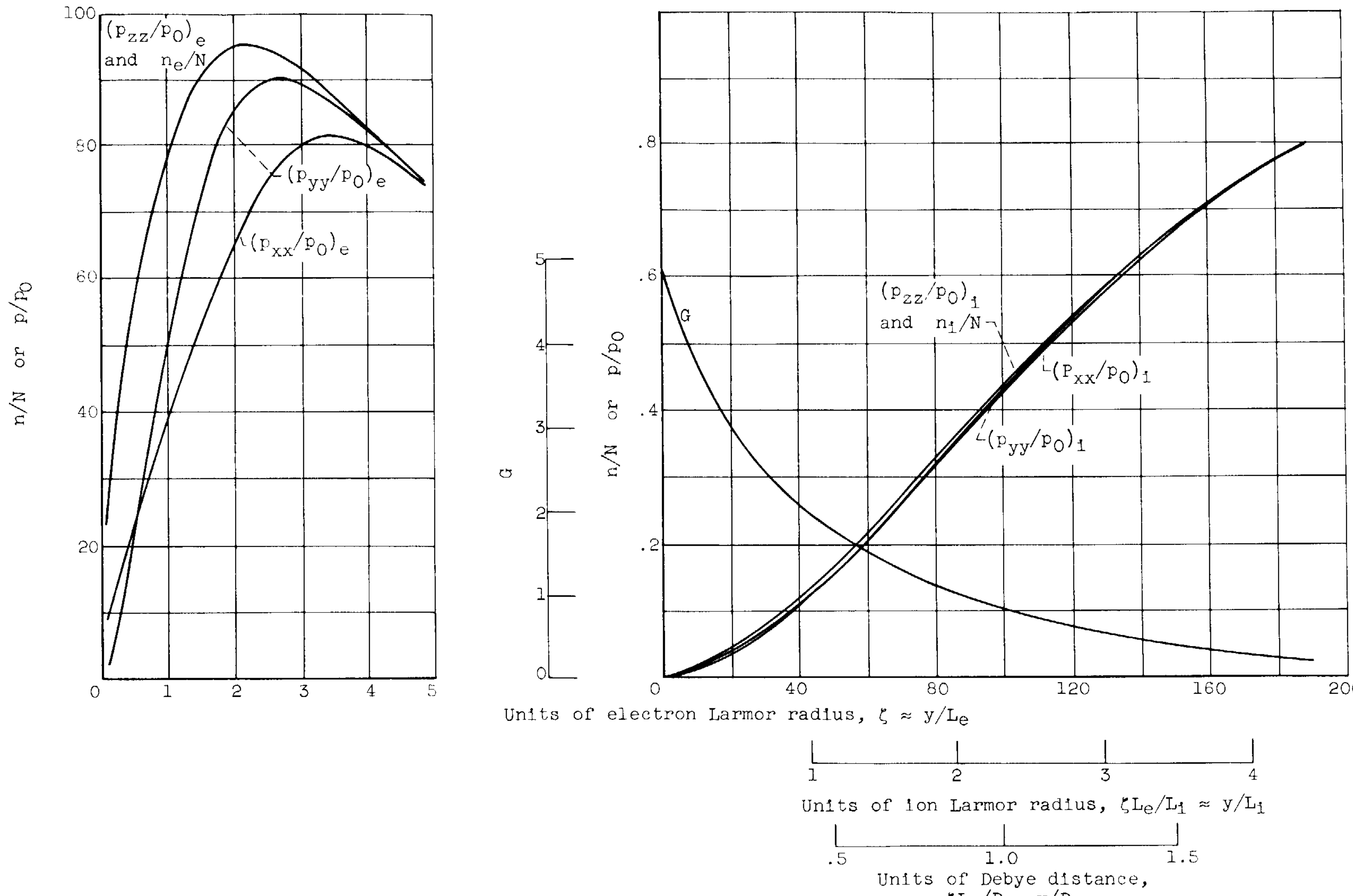


(a) ψ With no minimum.



(b) ψ With minimums.

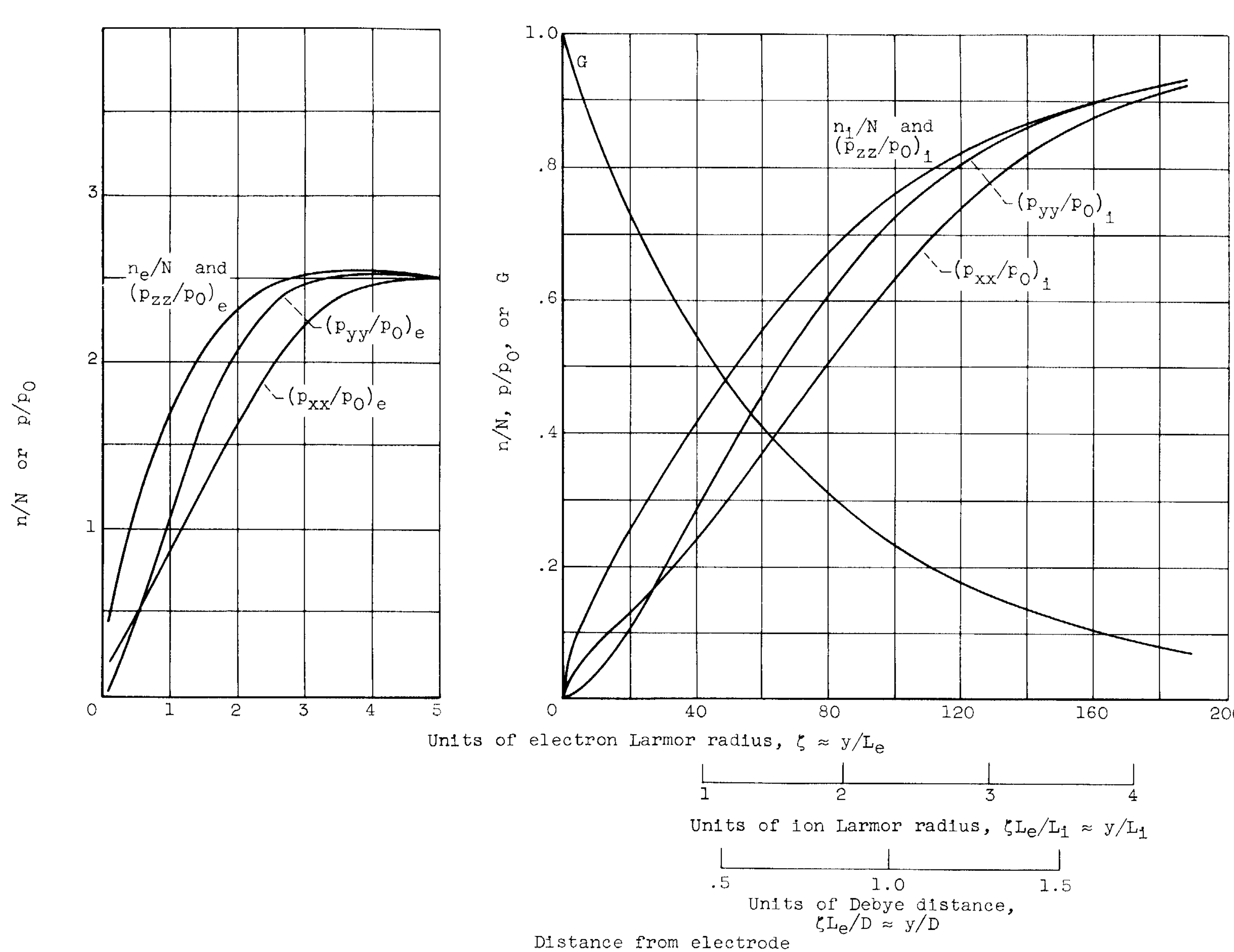
Figure 2. - Excursion range of particles.



Distance from electrode

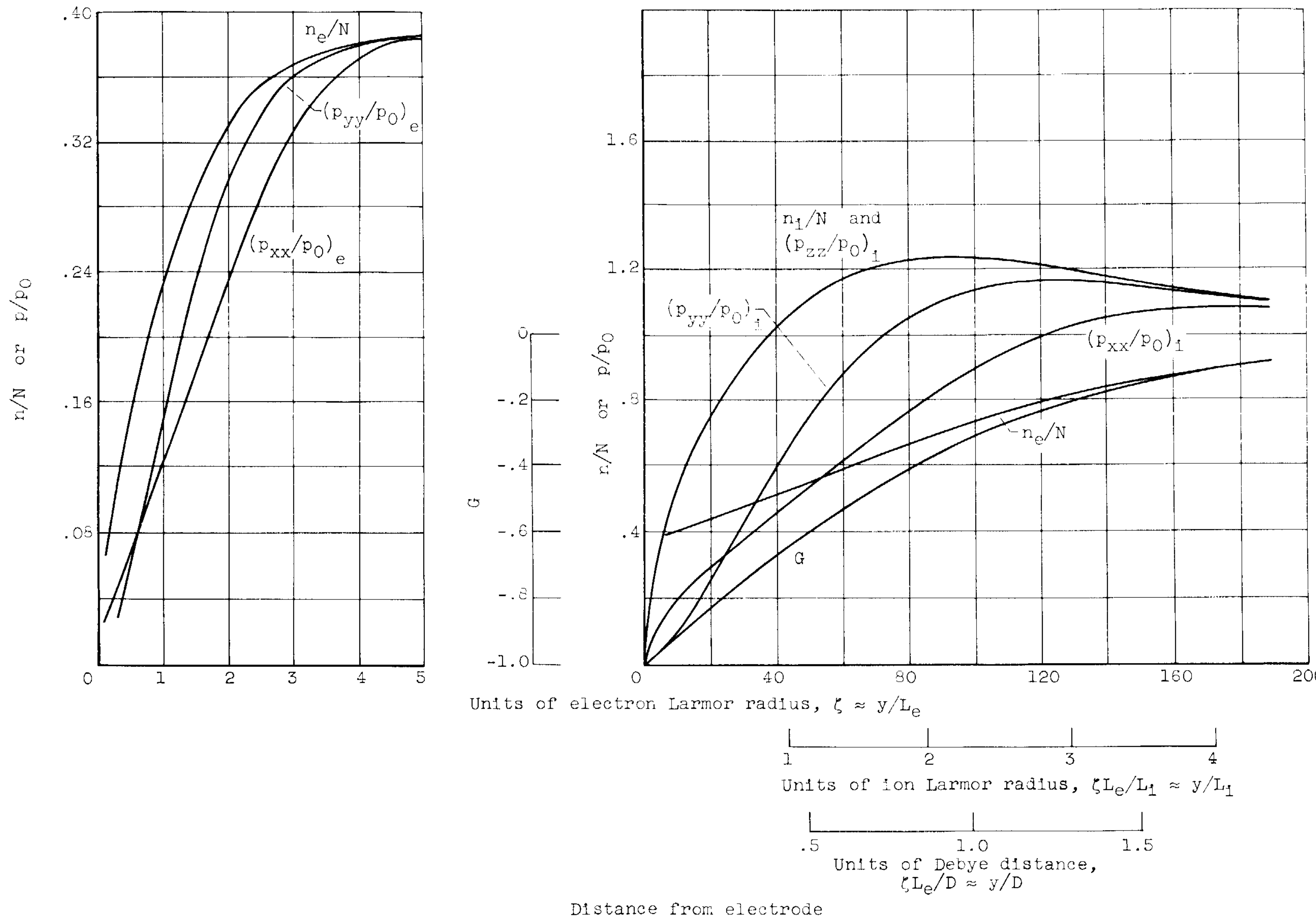
(a) $G_w = 5$; $L_e = 0.01D$.

Figure 3. - Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



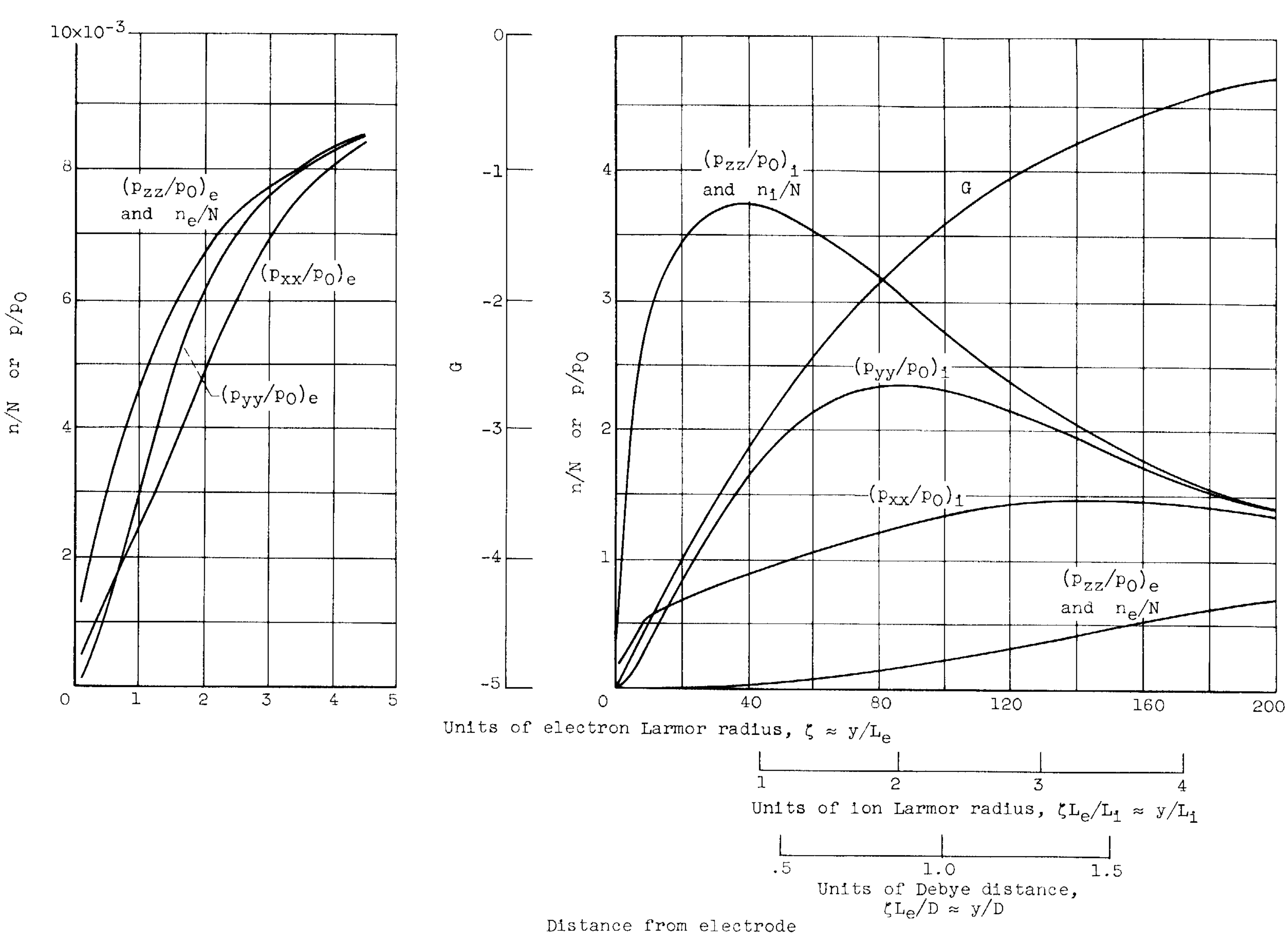
(b) $G_w = 1$; $L_e = 0.01D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



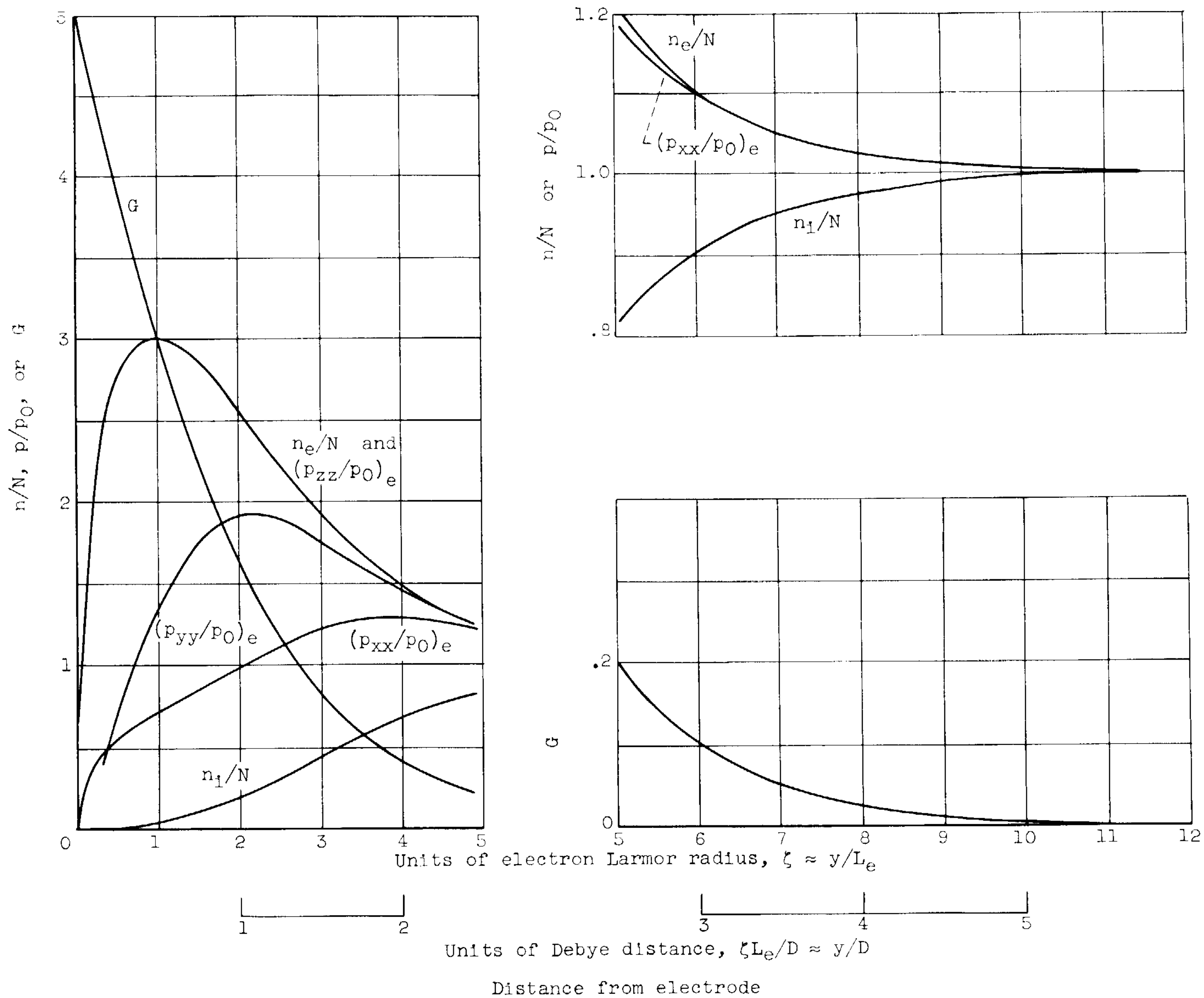
(c) $G_w = -1$; $L_e = 0.01D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



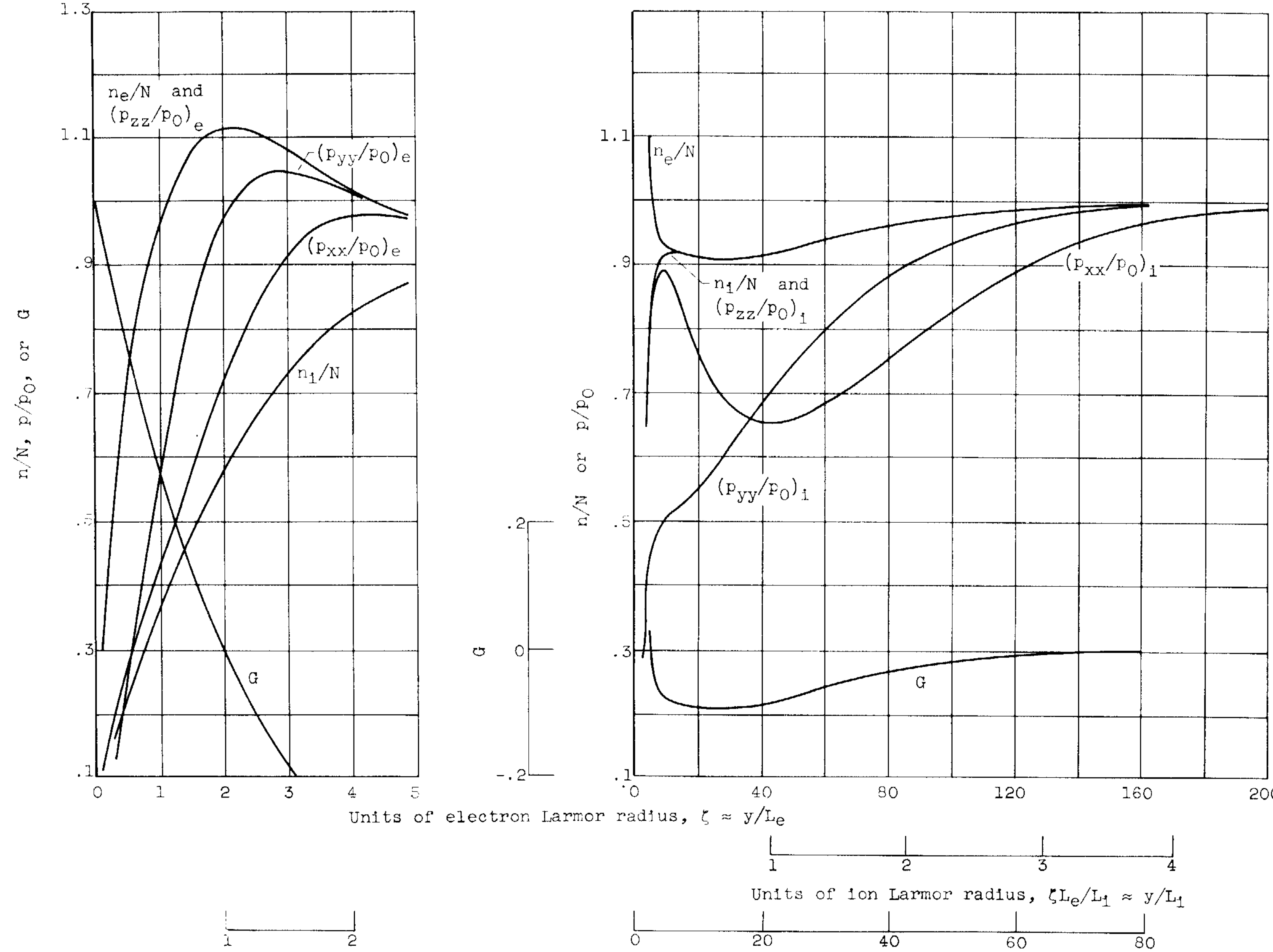
(d) $G_w = -5$; $L_e = 0.01D$; $a/c = 0$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



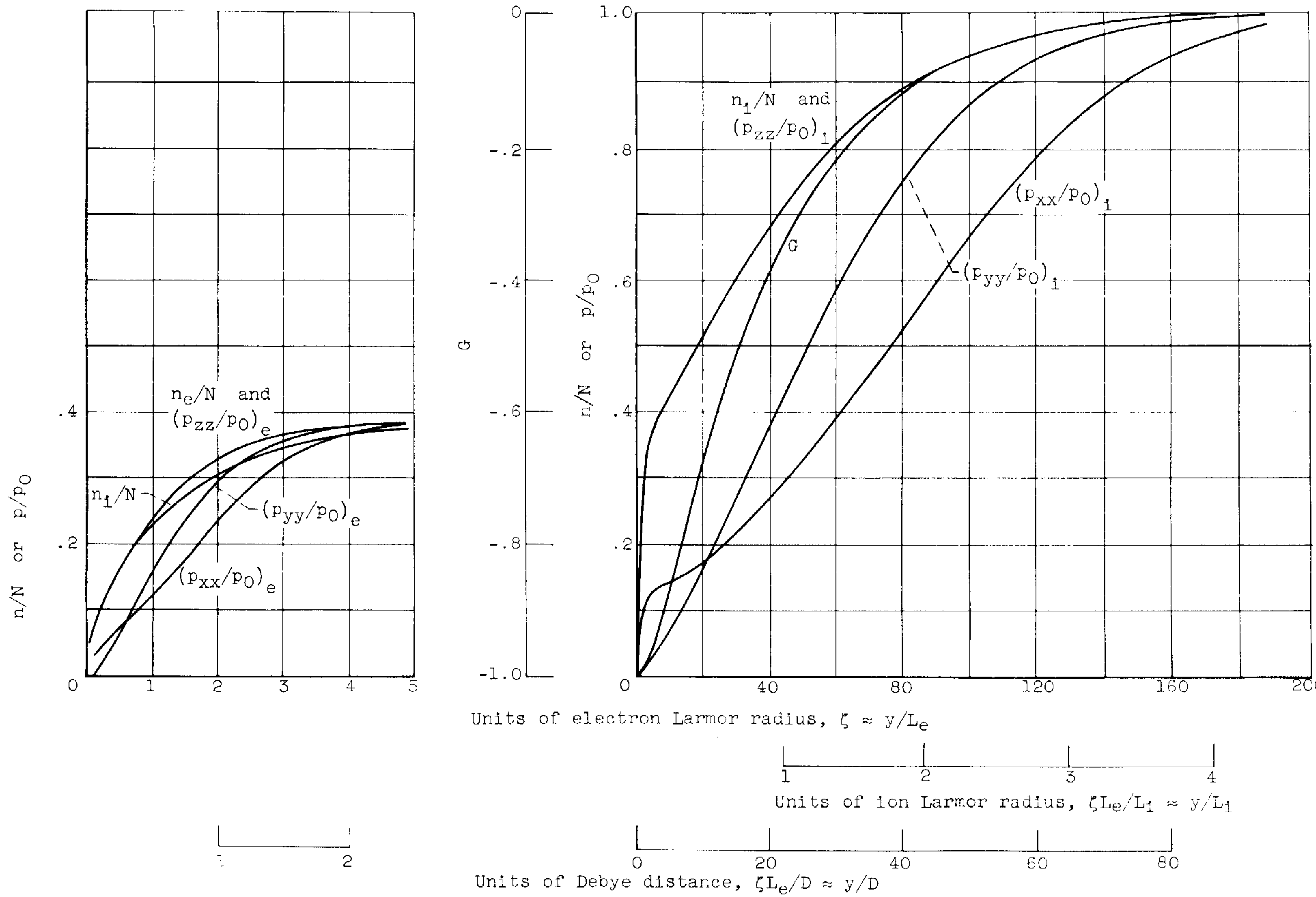
(e) $G_w = 5$; $L_e = 0.5D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



(f) $G_w = 1$; $L_e = 0.5D$.

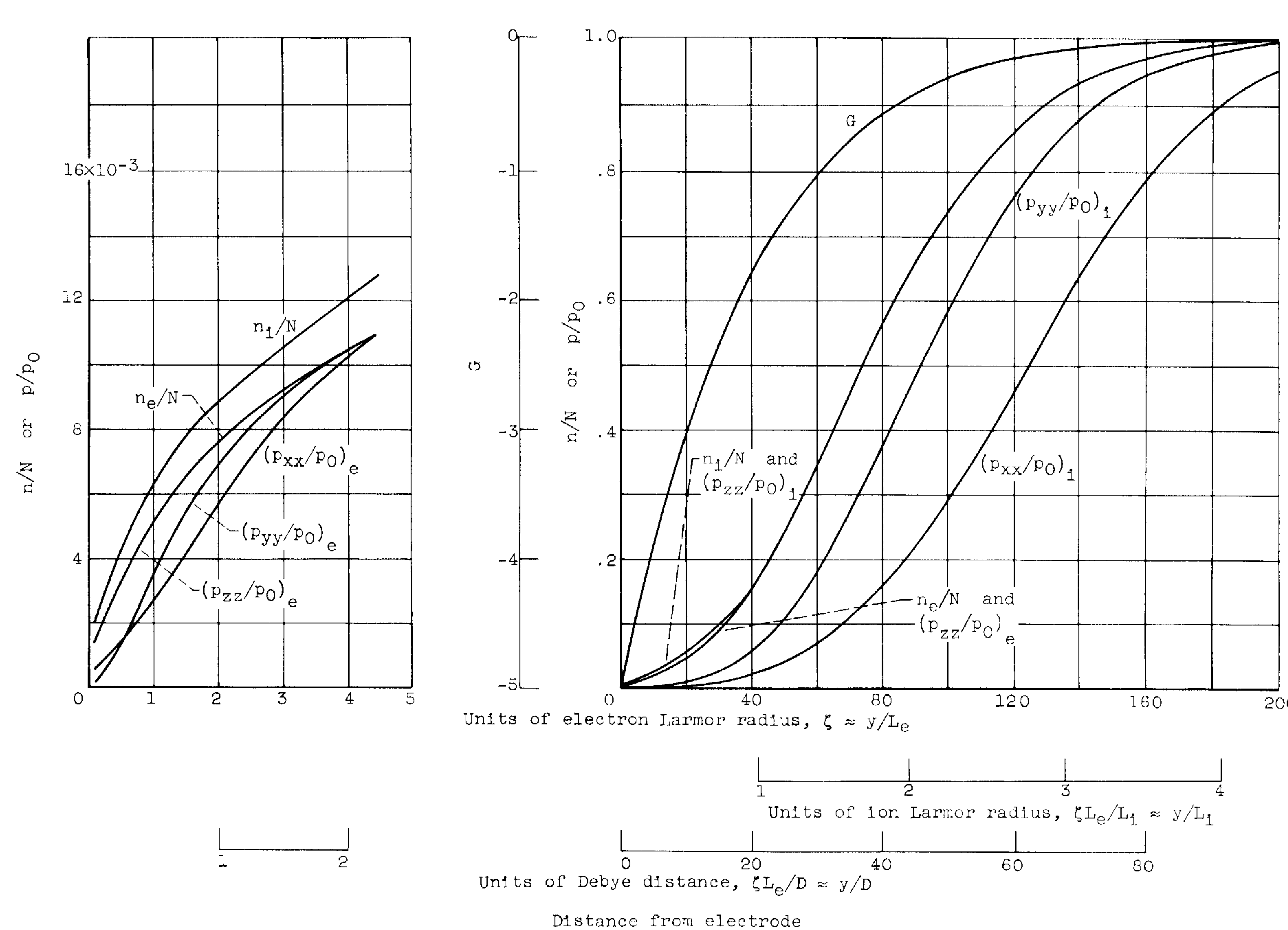
Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



Distance from electrode

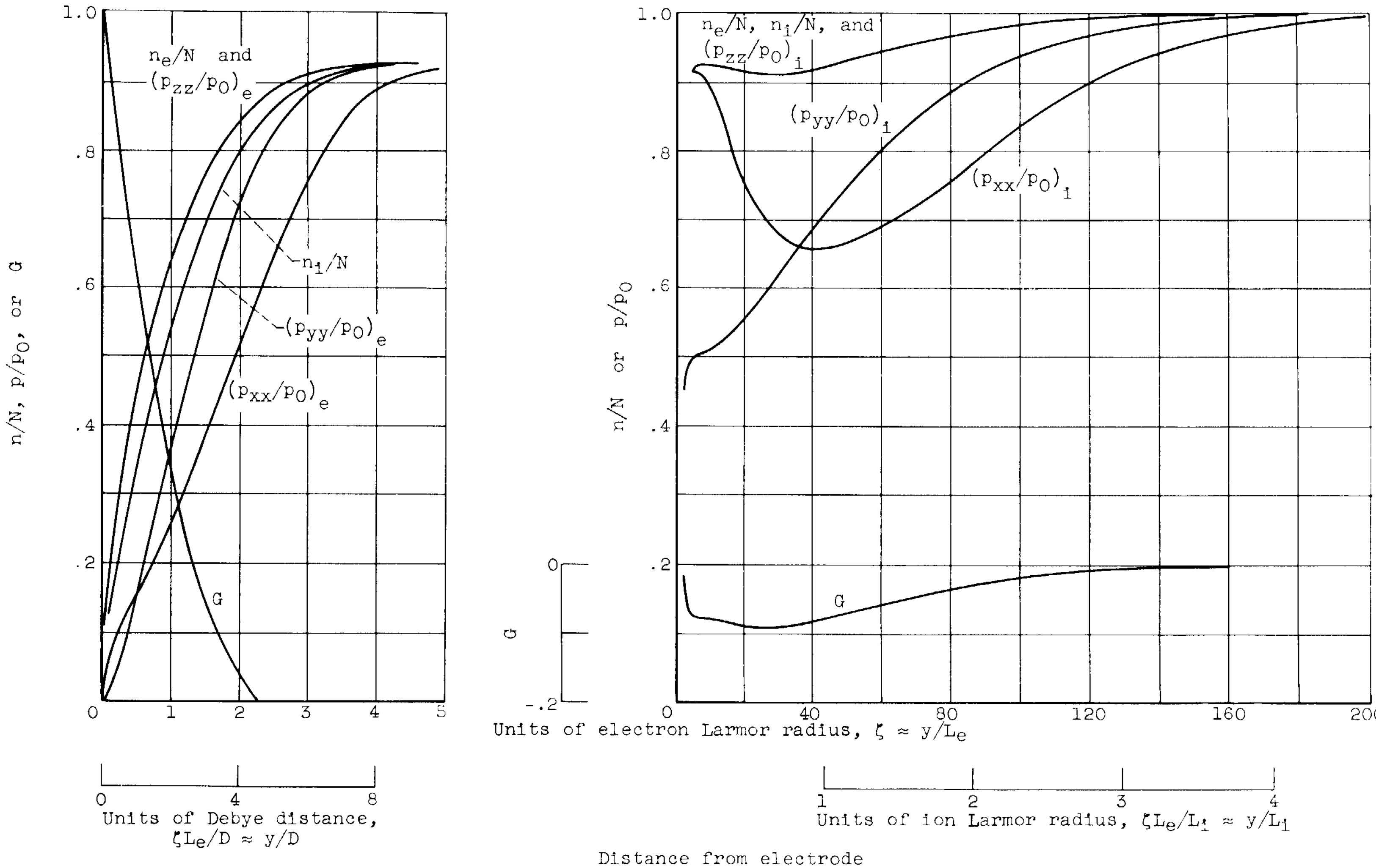
(g) $G_w = -1$; $L_e = 0.5D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



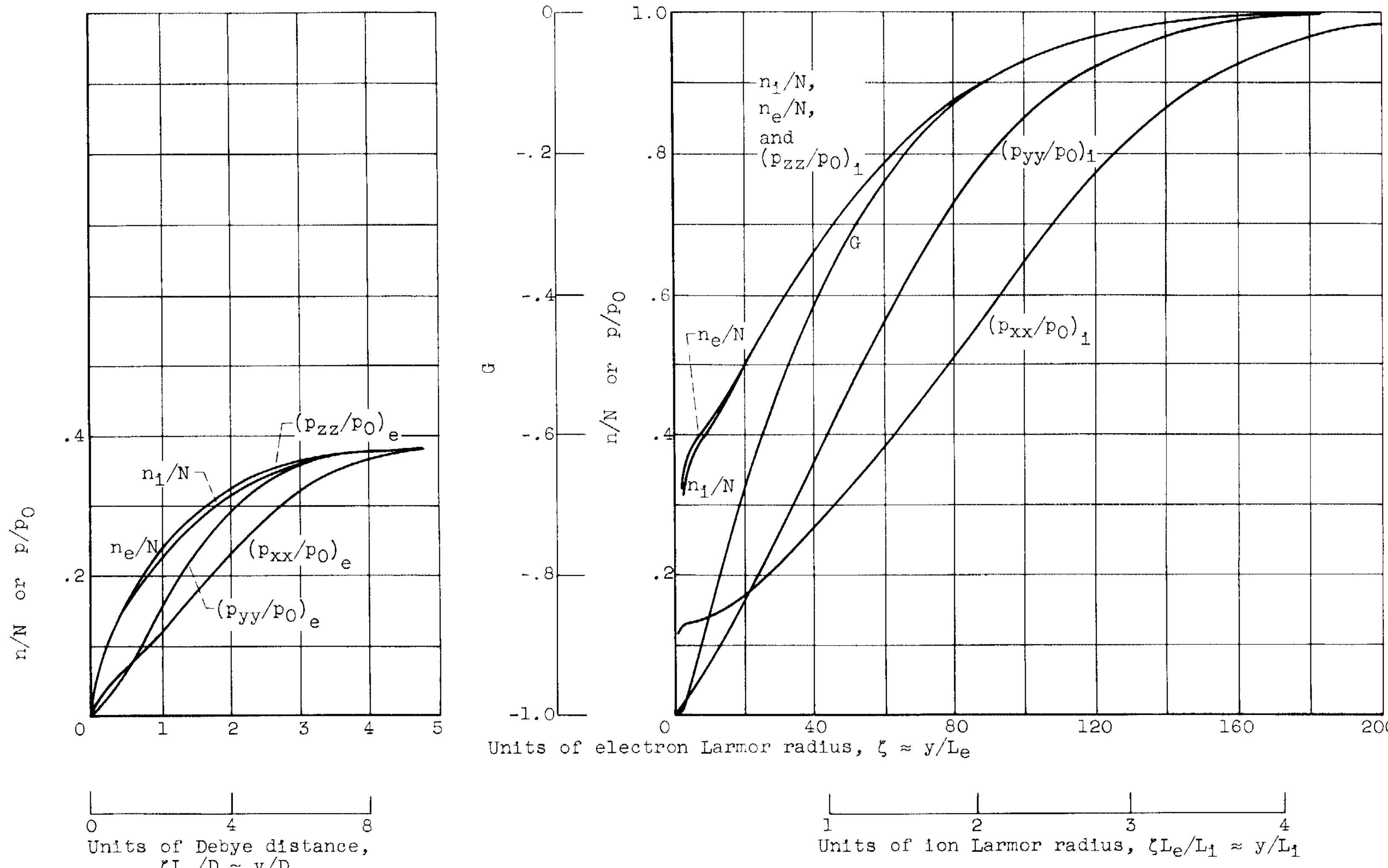
(h) $G_w = -5$; $L_e = 0.5D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



(i) $G_w = 1$; $L_e = 2D$.

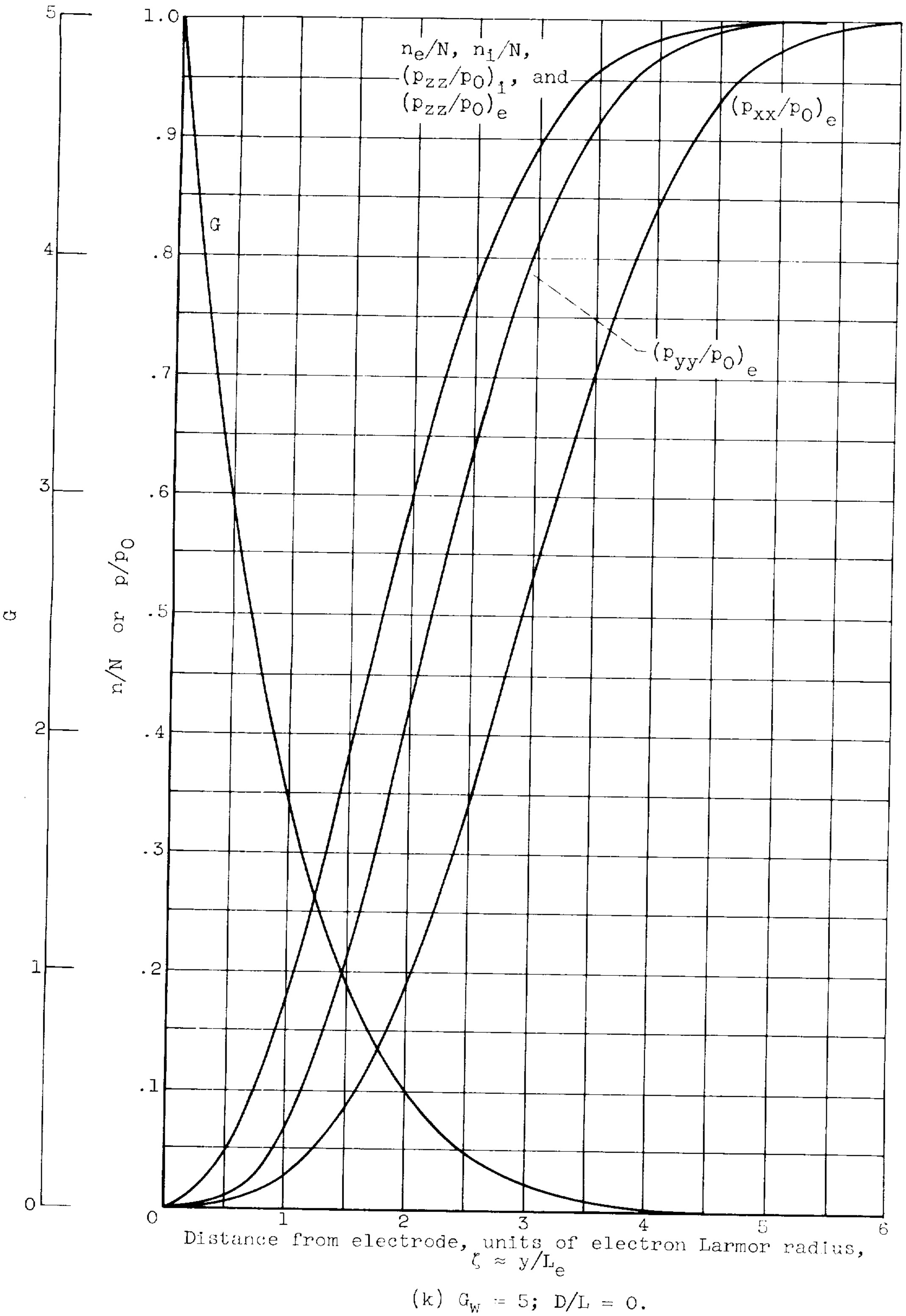
Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



Distance from electrode

(j) $G_w = -1$; $L_e = 2D$.

Figure 3. - Continued. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.



(k) $G_w = 5$; $D/L = 0$.

Figure 3. - Concluded. Sheath structure for hydrogen plasma. Temperature of electrons same as temperature of ions.